ASSIGNMENT SET - I

Department of Mathematics

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B.Sc Hon.(CBCS)

Mathematics: Semester-I

Paper Code: C2T

[Algebra]

Answer all the questions

Unit-1

1) If $x+iy$ moves on the straight line $3x+4y+5=0$, then find the minimum value of $|x+iy|$.
2) If $(1+i\tan\alpha)^{1+i\tan\beta}$ can have real value, then show that one of them is $(\sec\alpha)^{sec^2\beta}$.
3) If the complex numbers $z_{1,}$ z_2 andz_3 represent the three points P, Q, R and be such that $lz_1+mz_2+nz_3=0$ where $+m+n=0$, then show that the points P, Q, R lie on a straight line.
4) Apply Descartes' rule if signs to ascertain the minimum of complex roots of the equation $x^6+3x^2-2x-3=0$.
5) Find the sum of 99th powers of the roots of the equation $x^7-1=0$.
6) z is a variable complex number such that $|z|=2$. Show that the point $z+\frac{1}{z}$ lies on an ellipse.
7) If $= log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, where θ is real, prove that $\theta=ilog\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)$.
8) Show that the condition that the sum of two roots of the equation $x^4+mx^2+mx+p=0$ be equal to the product of the other two roots is $(2p-n)^2=(p-n)(p+m-n)^2$.
9) If $a_1, a_2, \dots, a_n$ be n real positive quantities then prove that $A.M.\geq G.M.\geq H.M.$
10) Solve the equation $x^3-3x^2-33x+847=0$ by Cardan's method.
11) State and prove Cauchy Schwarz's inequality.

12) Show that the equation

$(x-a)^3+(x-b)^3+(x-c)^3+(x-d)^3=0$

Where a, b, c, d are not all equal, has only one real root.

13) If $s_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$ prove that

(i) $s_n > \frac{2n}{n+1}$, if $n > 2$.

(ii) $\left(\frac{n-s_n}{n-1}\right)^{n-1} > \frac{1}{n}$, if $n > 2$.

14) If $x+\frac{1}{x}=2\cos\alpha$, $y+\frac{1}{y}=2\cos\beta$, $z+\frac{1}{z}=2\cos\gamma$, and $x+y+z=0$ then prove that

$\sum \sin 4\alpha = 2\sum \sin(\beta+\gamma)$

And $\sum \cos 4\alpha = 2\sum \cos(\beta+\gamma)$.

15) If a, b, c, x, y, z be all real numbers and $a^2+b^2+c^2=1$, $x^2+y^2+z^2=1$ then prove that $-1 \le ax+by+cz \le 1$.

If $a_1, a_2, \ldots a_n$ be n positive rational numbers and $s = a_1+a_2+\cdots+a_m$ prove that

$\left(\frac{s}{a_1}-1\right)^{a_1}\left(\frac{s}{a_2}-1\right)^{a_2}\ldots\left(\frac{s}{a_n}-1\right)^{a_n} \le (n-1)^s$.

16) Show that the solution of the equation $(1+x)^n-(1-x)^n=0$ are $x = i\tan\frac{\pi r}{n}$,

Where $r = 0, 1, 2, \ldots, n-1$, if n be odd

$= o, 1, 2, \ldots, \frac{n}{2}-1, \frac{n}{2}+1, \ldots, n-1$, if n be even.

17) Solve the equation $x^4-12x^3+47x^2-72x+36=0$

Given that the product of two of the roots is equal to the product of the other two.

18) Prove that the minimum value of $x^2+y^2+z^2$ is $\left(\frac{c}{7}\right)^2$ where x, y, z are positive real numbers subject to the condition $2x+3y+6z=c$, c being a constant. Find the values of x, y, z for which the minimum value is attained.

19) Solve the equation $3x^3-26x^2+52x-24=0$ given that the roots are in geometric progression.

20) If the equation $x^3+px^2+qx+r=0$ has a root $\alpha+i\alpha$ where p, q, r and α are real, prove that $(p^2-2q)(q^2-2pr)=r^2$.

Hence solve the equation $x^3-x^2-4x+24=0$.

21) State Descartes' rule of signs, Obtain the equation whose roots exceed the roots of the equation $x^4 + 3x^2 + 8x + 3 = 0$ by 1.
    Use Descartes' rule of signs to both the equations to find the exact number of real and complex roots of the given equation.
22) If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, then form the equation whose roots are $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}, \gamma + \frac{1}{\gamma}$.
23) Solve the equation $x^4 + 12x - 5 = 0$ by Ferrari's method.

Unit-2

24) If $f : A \longrightarrow B$ amd $g : B \longrightarrow C$ be two mapping such that $g \circ f : A \longrightarrow C$ is surjective , then show that g is surjective .
25) Use the 2nd principle of Induction to prove that $(3 + \sqrt{7})^n + (3 - \sqrt{7})^n$ is an even integer for all $n \epsilon \mathbb{N}$.
26) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined by $f(x) = x^2, x \epsilon \mathbb{R}$.
    and suppose $P = \{x \epsilon \mathbb{R} : 0 \leq x \leq 4\}$. Find $f^{-1}[f(p)]$. Is $f^{-1}[f(p)]$ equal to p.
27) Find $f \circ g$, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $f(x) = |x| + x, x \epsilon \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $g(x) = |x| - x,\ x \epsilon \mathbb{R}$.
28) Let $f : A \longrightarrow B$ and $P \subseteq A$. Then prove that $P \subseteq f^{-1}f(P)$.
29) Find the integer u and v satisfying $52u - 91v = 78$.
30) Using the principle of induction, prove that $2.7^n + 3.5^n - 5$ is divisible by 24 for $n \epsilon \mathbb{N}$.
31) Find the remainder when $1! + 2! + 3! + \cdots + 50!$ is divided by 15.
32) Prove that intersection of two equivalence relation is also an equivalence relation.
33) Examine if the relation ρ on the set $\mathbb{Z}$ is an equivalence relation or not
    $\rho = \{(a, b) \epsilon \mathbb{Z} \times \mathbb{Z} : |a - b| \leq 3\}$.
34) Prove that, there exists no integer in between 0 and 1.
35) Let $P = \{n \epsilon \mathbb{Z} : 0 \leq n \leq 5\}$, $Q = \{n \epsilon \mathbb{Z} : -5 \leq n \leq o\}$ be two sets. Prove that cardinality of two sets are equal.
36) Find the units digit in 7^{99}.

37) Use division algorithm to prove that the square of an odd integer is of the form $(8k+1)$, where k is an integer.
38) Define equivalence relation. A relation ρ is defined on $\mathbb{N}\times\mathbb{N}$ by "$(a,b)\rho(c,d)$ if and only if $ad-bc$" for $(a,b),(c,d)\epsilon\mathbb{N}\times\mathbb{N}$. Show that ρ is an equivalence relation.
39) If X and Y are two non-empty sets and $f:X\longrightarrow Y$ be an onto mapping, then for any subsets A and B of Y, prove that $f^{-1}(A\cup B)=f^{-1}(A)\cup f^{-1}(b)$.
40) Prove that $1^n-3^n-6^n+8^n$ is divisible by $10\forall n\epsilon\mathbb{N}$.
41) Find integers u and v satisfying $20u+63v=1$.
42) State the division algorithm on the set of integers.
43) Find integers s and t such that
$\gcd(341,1643)=341s+1643t$.
44) Using the theory of congruence for finding the remainder when the sum $1^5+2^5+3^5+\cdots+100^5$ is divided by 5.

45) Find the remainder when 777^{777} is divided by 16.
46) If k be $a+ve$ integer, show that $\gcd(ka,kb)=k\gcd(a,b)$.
47) Prove that tree is a one-to-one correspondence between the sets $(0,1)$ and $[0,1]$.
48) Use the theory of congruences to prove that $17|2^{3n+1}+3.5^{2n+1},\forall n\geq 1|$.
49) Let $S=\{x\epsilon\mathbb{R}:-1<x<1\}$ and $f:\mathbb{R}\longrightarrow S$ be defined by $f(x)=\frac{x}{1+|x|},x\epsilon\mathbb{R}$. Show that f is a bijection and find f^{-1}.

Unit-3

50) Solve the system of equations:
$x+2y-z-3w=1$
$2x+4y+3z+w=3$
$3x+6y+4z-2w=5$
If possible.
51) For what values of k the system of equations
$x+2y+3z=kx$
$2x+y+3z=ky$
$2x+3y+z=kz$
Has a non trivial solution.

52) Obtain the fully row reduced normal form of the matrix: $\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$.

53) For what values of k, the planes $x - 4y + 5z = k$, $x - y + 2z = 3$, $2x + y + z = 0$ intersect in a line.

54) Find a row echelon matrix which is row equivalent to $\begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$.

55) Show that the planes $2x - y + z = 5$, $x + 2y + 4z = 7$, $5x + 3y - z = 0$ are concurrent.

56) Let x, y, z be elements of a vector space V over F and. Let $a, b \epsilon F$. Show that x, y, z are linearly dependent, if $(x + ay + bz), y, z$ be linearly dependent.

57) Find the condition on $a, b \epsilon \mathbb{R}$ so that the set $\{(a, b, 1), (b, 1, a), (1, a, b)\}$ is linearly dependent in $\mathbb{R}^3$.

58) Find all real λ for which the rank of matrix A is 2:

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 3 & \lambda \\ 1 & 1 & 6 & \lambda = 1 \end{pmatrix}.$$

59) Solve the system of equations

$x_2 + x_3 = a$

$x_1 + x_3 = b$

$x_1 + x_2 = c$ and use the solution to find the inverse of the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

60) Investigate, for what values of α and μ, the following equations

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

have i) No solution

ii) a unique solution

and iii) an infinite number of solutions.

Unit-4

61) Find the rank of the matrix: $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$

 If two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_ay + c_2 = 0$ are coincident.

62) Show that the rank of a skew symmetric matrix cannot be 1.

63) Stay Cayley-Hamilton theorem and using theorem find A^{-1}, where $A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

64) A and B are any two 2×2 matrix and E is the corresponding unit matrix. Show that $AB - BA = E$ cannot hold under any circumstance.

65) If λ be an eigen value of a non-singular matrix A then prove that λ^{-1} is an eigen value of A^{-1}.

66) A linear mapping $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3, x_1 - 3x_2 - 2x_3). (x_1, x_2, x_3)\epsilon\mathbb{R}^3$. Find the matrix of T relative to the ordered bases $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

67) Define an eigen vector of a matrix $A_{n\times n}$ over filed F. Show that there exist many eigen vectors of A corresponding to an eigen value $\lambda\epsilon F$.

68) Show that the intersection of two subspaces of a vector space over a filed F is a subspace of V.

69) If A be a real skew symmetric matrix of order n, prove that :

 (i) The matrix $S - I_n$ is non-singular,

 (ii)The matrix $(S - I_n)^{-1}(S + I_n)$ is orthogonal,

 (iii)If X be an eigen vector of S with eigen value λ, then X is also an eigen vector of $(S - I_n)^{-1}(S + I_n)$ with eigen value $\frac{\lambda+1}{\lambda-1}$.

70) Let $S = \{(x, y, z)\epsilon\mathbb{R}^3 : x + y + z = 0\}$.

 Prove that S is a subspace of $\mathbb{R}^3$. Find a basis of S. Determine the dimension of S.

71) Let $S = \{(x, y, z, w)\epsilon\mathbb{R}^3 : x + 2y - z = 0, 2x + y + w = 0\}$ prove that S is a subspace of the real vector space $\mathbb{R}^4$. Also find the basis of S and the dimension of S.

72) A is a 3×3 real matrix having the eigen values $2, 3, 1$. If $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are the eigen vectors of A corresponding to the eigen values $2, 3, 1$ respectively. Find the matrix A.

73) Let V be a vector space over a field F and let $\alpha, \beta \epsilon v$. Then prove that the set $w = \{c\alpha + d\beta : c\epsilon F, d\epsilon F\}$ form a subspace of V.
If $\alpha = (1, 2, 3), \beta = (3, 1, 0)$ and $\gamma = (2, 1, 3)$ then examine for $\gamma \epsilon\, w$ or not.

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